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FURTHER COMMENTS ON THE APPLICATION OF THE METHOD OF AVERAGING TO THE STUDY OF THE ROTATIONAL MOTIONS OF A TRIAXIAL RIGID BODY, PART 2

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FURTHER COMMENTS ON THE APPLICATION OF THE METHOD OF AVERAGING TO THE STUDY OF THE ROTATIONAL MOTIONS OF A TRIAXIAL RIGID BODY, PART 2

1. Introduction

In [A.R.,1971]*, we described some of the results which we have obtained in applying the averaging technique described in [F.R.,1971] to the variational equations which arise in treating the perturbations of the free rotational motions of a triaxial rigid body. In [A.R.,1971] we carried out the first step of the averaging procedure and derived the averaged differential equations for a set of canonical variables (α_k, β_k) for the problem of a triaxial body in a precessing, elliptic orbit about an attracting center. The development was carried out to the point that the averaged differential equations [A.R., 1971,(5.31)] are in a form which can readily be integrated if it is so desired. The second step of the averaging procedure was not carried out for these canonical variables (α_k,β_k) because we were planning to use the averaging technique to develop first-order secular solutions for an alternative set of noncanonical variables. In [F.R.,1972] we began our discussion of the development of these secular solutions by carrying out the first step of the averaging procedure for a convenient set of noncanonical variables. In the present report, we complete the second and final step in the development of these first - order secular solutions.

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* References to our earlier reports of June 16, 1970, February 19, 1971, August 2, 1971 and February 21, 1972 are indicated by [J.R.,1971], [F.R.,1971], [A.R.,1971] and [F.R.,1972], respectively.

2. First-Order. Secular Solution

Equations (4.24)[+] may be integrated numerically with a much longer integration step time than may be used with equations (4.3). In this way, we can obtain the first-order secular solutions for the rotational motion under the influence of the gravity-gradient torque. We can also, however, integrate the averaged system (4.24) analytically with the aid of the integral (4.30). We address ourselves to this problem in the remainder of this section.

To begin with, we attempt to integrate (4.24(b)), noting that $-s_{\theta_0}\dot{\Omega}$ is constant. We first express c_{ψ_H} (and hence s_{ψ_H}) in terms of θ_H by substituting (b) and (c) of (4.25) into (4.30). We find that, to first order,

$$c_{\psi_H} = -\frac{1}{s_{\theta_H}}\left(\frac{c''h}{2b''}c^2_{\theta_H} - \frac{a''}{b''}c_{\theta_H} + C_h\right) , \qquad (2.1)$$

It follows from (2.1) that

$$s_{\psi_H} = \pm\frac{(A^*)^{1/2}}{s_{\theta_H}}\left(-c^4_{\theta_H} + A^*_3 c^3_{\theta_H} + A^*_2 c^2_{\theta_H} + A^*_1 c_{\theta_H} + A^*_0\right)^{1/2}, \qquad (2.2)$$

where

$$A^* = \frac{c''^2h^2}{4b''^2} ,$$

+ All equations designated (4.ij), i, j, nonnegative integers, refer to equations given in [F.R., 1972].

$$A_0^* = \frac{4b''^2(1-C_h^2)}{c''^2h^2} \quad ,$$

$$A_1^* = -\frac{8a''b''C_h}{c''^2h^2} \quad ,$$

$$A_2^* = -\frac{4b''^2}{c''^2h^2}\left(\frac{a''^2}{b''^2} - \frac{c''hC_h}{b''} - 1\right) \quad ,$$

$$A_3^* = \frac{4a''}{c''h} \quad .$$

If (2.2) is substituted into (4.24(b)), we can write

$$\frac{dc_{\theta_H}}{(-c_{\theta_H}^4 + A_3^*c_{\theta_H}^3 + A_2^*c_{\theta_H}^2 + A_1^*c_{\theta_H} + A_0^*)^{1/2}} = \mp(A^*)^{1/2}s_{\theta^o}\dot{\Omega}\,dt. \qquad (2.3)$$

In order to integrate (2.3), we first note that the bi-quadratic equation

$$-c_{\theta_H}^4 + A_3^*c_{\theta_H}^3 + A_2^*c_{\theta_H}^2 + A_1^*c_{\theta_H} + A_0^* = 0 \qquad (2.4)$$

has the following allowable roots:

(i) four distinct real roots,

(ii) four real roots with two identical roots,

(iii) two distinct real roots with a pair of complex roots,

(iv) two identical real roots and a pair of complex roots.

In what follows, we consider each of the four cases.

Case (i): Four distinct real roots

In this case, we can write equation (2.3) in the Jacobian normal form for real roots [see (BF250.04) and (BF250.06)]‡

$$g^* du^* = \frac{dc_{\theta_H}}{[(a_1-c_{\theta_H})(a_2-c_{\theta_H})(a_3-c_{\theta_H})(c_{\theta_H}-a_4)]^{1/2}} = \mp(\Lambda^*)^{1/2} s_{\theta_o}\dot{\Omega}\, dt\ , \tag{2.5}$$

where

$$g^* = \frac{2}{[(a_1-a_3)(a_2-a_4)]^{1/2}}\ , \tag{2.5'}$$

if a_1, a_2, a_3 and a_4 are the real roots of the biquadratic equation (4.4), and it is assumed that $a_1 > a_2 > a_3 > a_4$. The variable c_{θ_H} is related to u^* through the equation

$$c_{\theta_H} = \frac{A_1 + A_2\ \mathrm{sn}^2\ u^*}{A_3 + A_4\ \mathrm{sn}^2\ u^*}\ , \quad 0 \leq u^* \leq K^*. \tag{2.6}$$

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‡ Reference to equations in <u>H and Book of Elliptic Integrals for Engineers and Physicists</u>, Byrd, P.F. and Friedman, M.D., Springer, Berlin, 1954, are prefixed by the notation BF.

Here K^{*} is defined by the integral

$$K^{*} = \int_{0}^{\pi/2} \frac{d\xi}{(1-k^{*2}\sin^{2}\xi)^{1/2}} \quad ,$$

where k^{*} is the modulus of Jacobian elliptic functions and integrals. Explicit values of k^{*} and A_i, $i=1,2,3,4$ will be given in the study of the special cases which follows.

Since the time rate of change of c_{θ_H} is real, the value of c_{θ_H} must either lie between a_1 and a_2 inclusive or between a_3 and a_4 inclusive. We will analyze (2.5) in the subcases which follow.

(1) $a_1 > c_{\theta_H} > a_2$: The A_i, $i=1,2,3,4$, have the values

$$\begin{aligned}
A_1 &= a_1(a_2 - a_4) \quad , && \text{(a)} \\
A_2 &= a_4(a_1 - a_2) \quad , && \text{(b)} \\
A_3 &= a_2 - a_4 \quad , && \text{(c)} \\
A_4 &= a_1 - a_2 \quad , && \text{(d)}
\end{aligned} \tag{2.7}$$

and k^{*2} has the value

$$k^{*2} = \frac{(a_1 - a_2)(a_3 - a_4)}{(a_1 - a_3)(a_2 - a_4)} \tag{2.8}$$

for this and the remaining three possiblities under Case (i). If we integrate (2.5) with respect to u^* from 0 to u^* (i.e., integrating with respect to time t from t_1 to t), we find [see (BF257.00)] that

$$u_1^* = \pm \frac{1}{g^*} (A^*)^{1/2} s_{\theta o} \dot{\Omega} (t-t_1), \tag{2.9}$$

where t_1 is the value of t at which $c_{\theta_H} = a_1$.

Substituting (2.9) into equation (2.6), we obtain the first-order secular solution for c_{θ_H} and thus the first-order secular solution for θ_H.

(2) $a_1 \geq c_{\theta_H} > a_2$: The A_i, i=1,2,3,4, have the values

$$\begin{aligned} A_1 &= a_2(a_3 - a_1) , && \text{(a)} \\ A_1 &= a_3(a_1 - a_2) , && \text{(b)} \\ A_3 &= a_3 - a_1 , && \text{(c)} \\ A_4 &= a_1 - a_2 . && \text{(d)} \end{aligned} \tag{2.10}$$

If we integrate (2.5) with respect to time from t_2 to t, we find [see(BF256.00)] that

$$u_2^* = \mp \frac{1}{g^*} (A^*)^{1/2} s_{\theta o} \mathring{\Omega} (t-t_2) , \tag{2.11}$$

where t_2 is the value of t at which $c_{\theta_H} = a_2$.

Substituting (2.11) into equation (2.6), we obtain the first-order secular solution for θ_H.

(3) $a_3 < c_{\theta_H} < a_4$: The A_i, i=1,2,3,4, have the values

$$
\left.
\begin{aligned}
A_1 &= a_3(a_4 - a_2) \quad , && \text{(a)}\\
A_2 &= a_2(a_3 - a_4) \quad , && \text{(b)}\\
A_3 &= a_4 - a_2 \quad , && \text{(c)}\\
A_4 &= a_3 - a_4 \quad , && \text{(d)}
\end{aligned}
\right\} \quad (2.12)
$$

If we integrate (2.5) with respect to time t from t_3 to t, we have [see (BF253.00)] that

$$u_3^* = \pm \frac{1}{g^*} (A^*)^{1/2} s_{\theta o}\dot{\Omega}(t-t_3) \quad , \tag{2.13}$$

where t_3 is the value of t at which $c_{\theta_H} = a_3$

Substituting (2.15) into equation (2.6), we obtain the first-order secular solution for θ_H.

(4) $a_3 \leq c_{\theta_H} < a_4$: The A_i, i=1,2,3,4, have the values

$$
\left.
\begin{aligned}
A_1 &= a_4(a_1 - a_3) \quad , && \text{(a)}\\
A_2 &= a_1(a_3 - a_4) \quad , && \text{(b)}\\
A_3 &= a_1 - a_3 \quad , && \text{(c)}\\
A_4 &= a_3 - a_4 \quad , && \text{(d)}
\end{aligned}
\right\} \quad (2.14)
$$

If we integrate (2.5) with respect to time t from t_4 to t, we find [see(BF252.00)] that

$$u_4^* = \mp \frac{1}{g^*} (A^*)^{1/2} s_{\theta^\circ} \mathring{\Omega} (t - t_4) \quad , \qquad (2.15)$$

where t_4 is the value of t at which $c_{\theta_H} = a_4$.

Substituting (2.15) into equation (2.6), we obtain the first-order secular solution for θ_H.

The associated first-order secular solution for the variable ψ_H for each of the subcases described above, is readily obtained from (2.1). This eliminates the need to integrate (4.24(a)) directly. Four more integrals of (4.24) remain to be determined. We consider next the variable ϕ_H.

In order to integrate equation (4.24(c)), we first rewrite it in the form

$$\dot{x}_3 = (s_{\theta^\circ} \mathring{\Omega}) \frac{c_{\psi_H}}{s_{\theta_H}} - (x_1'' - 3x_3'') c^2_{\theta_H} - x_3'' \quad . \qquad (2.16)$$

We note that if we can integrate c_{ψ_H}/s_{θ_H} and $c^2_{\theta_H}$ with respect to time, to first-order, we can then obtain the solution for ϕ_H.

From relations (4.25(b)) and (4.30), it is found, to first-order, that

$$\frac{c_{\psi_H}}{s_{\theta_H}} = - \frac{h_{y^\circ}}{h s^2_{\theta_H}} = - \frac{h_{y^\circ}}{h - h c^2_{\theta_H}} = - \frac{h\left(\frac{c''}{2b''} h^2_{z^\circ} - \frac{a''}{b''} h_{z^\circ} - C_h\right)}{h^2 - h^2_{z^\circ}} \quad . \qquad (2.17)$$

If we introduce the identities

$$\frac{1}{h^2-h_{z^\circ}^2} = \frac{1}{2h}\left(\frac{1}{h+h_{z^\circ}} + \frac{1}{h-h_{z^\circ}}\right), \qquad (a)$$

$$\frac{h_{z^\circ}}{h^2-h_{z^\circ}^2} = \frac{1}{2}\left(\frac{1}{h-h_{z^\circ}} - \frac{1}{h+h_{z^\circ}}\right), \qquad (b) \qquad (2.18)$$

$$\frac{h_{z^\circ}^2}{h^2-h_{z^\circ}^2} = \frac{h}{2}\left(\frac{1}{h+h_{z^\circ}} + \frac{1}{h-h_{z^\circ}}\right)-1, \qquad (c)$$

equation (2.17) takes the separated form

$$\frac{c_{\psi_H}}{s_{\theta_H}} = \frac{D_1}{h}\frac{1}{h-c_{\theta_H}} + \frac{D_2}{h}\frac{1}{h+c_{\theta_H}} + D_3 , \qquad (2.19)$$

where

$$D_1 = -\frac{1}{2}(a''h^2 - b''h + c'') ,$$

$$D_2 = -\frac{1}{2}(a''h^2 + b''h + c'') ,$$

$$D_3 = a''h .$$

We can express the right hand side of (2.19) in terms of u^* if we replace c_{θ_H} through the use of (2.6). We use (2.5) to replace t by u^* as the variable of integration. We then obtain the differential form

$$\frac{c_{\psi_H}}{s_{\theta_H}}\,dt = \mp \frac{g^*}{(A^*)^{1/2} s_{\theta^\circ}\,\Omega}\left[\frac{D_1}{h(A_3-A_1)}\;\frac{A_3+A_4\ \mathrm{sn}^2u^*}{1-\gamma_1^2\ \mathrm{sn}^2u^*} +\right.$$

$$\left. + \frac{D_2}{h(A_3+A_1)}\;\frac{A_3+A_4\ \mathrm{sn}^2\,u^*}{1-\gamma_2^2\ \mathrm{sn}^2\,u^*} + D_3\right] du \;. \qquad (2.20)$$

where

$$-\gamma_1^2 = \frac{A_4 - A_2}{A_3 - A_1}\;, \qquad (a)$$

$$(2.20)'$$

$$-\gamma_2^2 = \frac{A_4 + A_2}{A_3 + A_1}\;. \qquad (b)$$

Upon integrating (2.20) from t_i to t, we find, from (BF336.01) and (BF337.01), that

$$I_{1i} = \int_{t_i}^{t} \frac{c_{\psi_H}}{s_{\theta_H}}\,dt = \mp \frac{g^*}{(A^*)^{1/2} s_{\theta^\circ}\,\Omega}\left\{\frac{D_1}{h(A_3-A_1)}\left[A_3\left(V_1(u_i^*,\gamma_1^2) - V_1(0,\gamma_1^2\right)\right.\right.$$

$$\left.\left. - A_4\left(W_1(u_i^*,\gamma_1^2) - W_1(0,\gamma_1^2)\right)\right] \qquad (2.21)$$

$$+ \frac{D_2}{h(A_3+A_1)}\left[A_3\left(V_1(u_i^*,\gamma_2^2) - V_1(0,\gamma_2^2)\right)\right.$$

$$\left.\left. - A_4\left(W_1(u_i^*,\gamma_2^2) - W_1(0,\gamma_2^2)\right)\right] - D_3\,u_i^*\right\}$$

where $i=1,2,3,4$, and

$$V_1(u_i^*, \gamma_j^2) = II(u_i^*, \gamma_j^2) \qquad (a)$$

$$(j=1,2) \qquad (2.22)$$

$$W_1(u_i^*, \gamma_j^2) = \frac{1}{\gamma_i^2}\left[II(u_i^*, \gamma_j^2) - F(u_i^*)\right] \quad ,(b)$$

Here $II(u_i^*, \gamma_j^2)$, j, a positive integer is Legendre's incomplete integral of the third kind and $\gamma_j^2 \neq 1$ or $\gamma_j^2 \neq k_i^{*2}$. For the special cases where $\gamma_j^2 = 1$ or $\gamma_j^2 = k^{*2}$, the reader may refer to (BF111.06) for appropriate formulas. It can be evaluated by using Formulas (BF430) through (BF440).

If, next, we square (2.6), we have that

$$c_{\theta_H}^2 = \frac{A_1^2}{A_3^2}\left(\frac{1-\gamma_3^2 \operatorname{sn}^2 u^*}{1-\gamma_4^2 \operatorname{sn}^2 u^*}\right)^2 \qquad (2.23)$$

where

$$-\gamma_3^2 = \frac{A_2}{A_1} \quad , \quad -\gamma_4^2 = \frac{A_4}{A_3} \qquad (2.24)$$

Repeating the procedure used to obtain I_{1i}, we find that (use BF340.02)

$$I_{2i} = \int_{t_i}^{t} c_{\theta_H}^2 dt = -\frac{g^*}{(A^*)^{1/2} s_{\theta o}\Omega} \frac{A_1^2}{A_3^2} \frac{1}{\gamma_4^4}\Big\{\gamma_3^4 u_i^* +$$

$$+ 2\gamma_3^2(\gamma_4^2 - \gamma_3^2)\left[V_1(u_i^*, \gamma_4^2) - V_1(0, \gamma_4^2)\right] \qquad (2.25)$$

$$+ (\gamma_4^2 - \gamma_3^2)^2 \left[V_2(u_i^*, \gamma_4^2) - V_2(0, \gamma_4^2) \right] \Big\} \quad (2.25)$$

where

$$V_1(u_i^*, \gamma_4^2) = \Pi(u_i^*, \gamma_4^2) \ , \qquad \text{(a)}$$

$$V_2(u_i^*, \gamma_4^2) = \frac{1}{2(\gamma_4^2-1)(k^{*2}-\gamma_4^2)} \Bigg[\gamma_4^2 E(u_i^*) + (k^{*2}-\gamma_4^2) u_i^*$$

$$+ (2\gamma_4^2 k^{*2} + 2\gamma_4^2 - \gamma_4^4 - 3k^{*2}) \ \Pi(u_i^*, \gamma_4^2) \qquad (2.26)$$

$$- \frac{\gamma_4^4 \operatorname{sn} u_i^* \operatorname{cn} u_i^* \operatorname{dn} u_i^*}{1-\gamma_4^2 \operatorname{sn}^2 u_i^*} \Bigg] \ . \qquad \text{(b)}$$

If we take the unperturbed solution $(\phi_H)_o$ as our initial value of ϕ_H, the initial value of x_3 will be zero and we can write

$$x_3 = \phi_H - (\phi_H)_o = (s_{\theta o} \dot{\Omega}) I_{1i} - (x_1'' - 3x_3'') I_{2i} - x_3'' t \ . \qquad (2.27)$$

The remaining three integrals of (4.24) follow easily. They are explicitly

$$x_4 = \theta' - (\theta')_o = 0 \qquad (2.28)$$

$$x_5 = \phi' - (\phi')_o = - x_5''(t - 3I_{2i}) \qquad (2.29)$$

$$x_6 = h - (h)_o = 0 . \qquad (2.30)$$

while the unperturbed solutions are taken as initial values of the relevant variables.

Summarizing, we have the secular, first-order solution for Case (i):

$$c_{\gamma_H} = - \frac{1}{s_{\theta_H}}\left(\frac{c'h}{2b'} \; c^2_{\theta_H} - \frac{a'}{b'} \; c_{\theta_H} + C_h \right) , \qquad (a)$$

$$c_{\theta_H} = \frac{A_1 + A_2 sn^2 \; u_i^*}{A_3 + A_4 sn^2 \; u_i^*} , \qquad 0 \le u_i^* \le K^* \qquad (b)$$

(2.31)

$$\phi_H = (\phi_H)_o + (s_{\theta o} \dot{\Omega}) I_{1i} - (x_1'' - 3x_3'') I_{2i} - x_3'' t , \quad (c)$$

$$\theta' = (\theta')_o , \qquad (d)$$

$$\phi' = (\phi')_o - x_5''(t - 3I_{2i}) , \qquad (e)$$

$$h = (h)_o , \qquad (f)$$

where u_i^* is given by whichever of (2.9),(2.11),(2.13) or (2.15) applies to the appropriate subcase.

Case(ii): Four real roots with identical roots

This case can be treated as a special case of (i) and can be further grouped into two subcases.

(1) $a_1 = a_2 \neq c_{\theta_H}$ or $a_3 = a_4 \neq c_{\theta_H}$: It is seen from (2.8) that $k^* = 0$. Thus all the elliptic functions reduce to trigonometric functions (i.e., sn $u^* = \sin u^*$, etc.). In either case, equation (2.6) becomes

$$c_{\theta_H} = \frac{A_1 + A_2 \sin^2 u_i^*}{A_3 + A_4 \sin^2 u_i^*}, \qquad 0 \leq u_i^* \leq \pi/2 , \qquad (2.32)$$

where the A_i, i=1,2,3,4, are given by either (2.12) or (2.14) if $a_1 = a_2$. They are given by either (2.7) or (2,10) if $a_3 = a_4$. Then u_i^* is given by (2.13) or (2.15) if $a_1 = a_2$ and it is given by (2.9) or (2.11) if $a_3 = a_4$.

We note that if we replace (2.31(b)) by (2.32) and if, for this case ($k^* = 0$), we can evaluate I_{1i} and I_{2i} which correspond to (2.21) and (2.25). Then equations (2.31) will give us the first-order secular solutions. It is also seen from equations (2.21) ,(2.22), (2.25) and (2.26) that if we evaluate both $II(u_i^*, \gamma_j^2)$, j=1,2,3,4 and $F(u_i^*)$ at $k^* = 0$ then these equations will determine both I_{1i} and I_{2i}. Formulas for the elliptic integrals $II(u_i^*, \gamma_j^2)$ and $F(u_i^*)$ can be found in [1]. Explicitly, they are [see (BF111.01) and (BF121.01)].

$$F(u_i^*) = u_i^*,$$

$$II(u_i^*, \gamma_j^2) = u_i^*, \quad \text{if} \quad \gamma_j^2 = 0 \qquad (2.33)$$

$$= \frac{\tan^{-1}[(1-\gamma_j^2)^{1/2}\tan u_i^*]}{(1-\gamma_j^2)^{1/2}}, \text{ if } \gamma_j^2 < 1,$$

(2.33)

$$= \frac{\tan^{-1}[(\gamma_j^2-1)^{1/2}\tan u_i^*]}{(\gamma_j^2-1)^{1/2}}, \quad \text{if } \gamma_j^2 > 1.$$

If $\gamma_j^2 = 1$, we can use (BF111.01), (BF121.01) and (BF111.06), and write

$$II(u_i^*, 1) = \tan u_i^* . \qquad (2.34)$$

Thus, the integrals I_{1i} and I_{2i} are determined and equations (a),(b),(c),(d),(e) and (f) of (2.31), together with (2.32), give the first-order secular solutions for the six variables of interest.

(2) $a_2 = a_3$: It follows from (2.8) that $k^* = 1$. Thus all elliptic functions reduce to hyperbolic functions (i.e., sn u^* = tan h u^*, cn u^* = sec h u^*). Equation (2.6) takes the form

$$c_{\theta_H} = \frac{A_1 + A_2 \tan h^2 u_i^*}{A_3 + A_4 \tan h^2 u_i^*}, \quad 0 \leq u_i^* \leq \infty \qquad (2.35)$$

where the A_i, i=1,2,3,4, are given by either (2.7), (2.10), (2.12) or (2.14) and u_i^* is given by the associated relation (2.9), (2.11), (2.13) or (2.15).

The analysis proceeds as in the preceding subcase, and we have, from (BF111.04), that if $k^* = 1$

$$F(u_i^*) = \ln(\tan\phi_i - \sec\phi_i), \quad \phi_i = \text{am } u_i^* \quad (a)$$

(2.36)

$$II(u_i^*, \gamma_j^2) = \frac{1}{1-\gamma_j^2}\left[\ln(\tan\phi_i + \sec\phi_i) - \gamma_j \ln\left(\frac{1+\gamma_j s_{\phi_i}}{1-\gamma_j s_{\phi_i}}\right)^{1/2}\right], \quad (\gamma_j^2 \neq 1). \quad (b)$$

[Here γ_j^2 cannot take on the value one since $II(u_i^*,1) = \infty$]. Thus the integrals I_{1i} and I_{2i} can be determined by (2.21), (2.22) and (2.25),(2.26), respectively, and therefore equations (a),(b),(c),(d),(e),(f) coupled with (2.35), give the first-order secular solutions.

Case (iii): Two distinct real roots and a pair of complex roots

Let a_1, a_2 be the real roots and let a_3, and its complex conjugate a_3^* be the complex roots and assume that $a_1 > a_2$. We can write equation (2.3) in the Jacobian normal form for complex roots [see (BF250.05) and (BF250.06)]

$$g^* du^* = \frac{dc_{\theta_H}}{[(a_1-c_{\theta_H})(c_{\theta_H}-a_2)(c_{\theta_H}-a_3)(c_{\theta_H}-a_3^*)]^{1/2}} = \mp (A^*)^{1/2} s_{\theta^\circ} \dot{\Omega}\, dt.$$

(2.37)

where g^* is given by the equation

$$g^* = \frac{1}{(A' \ B')^{1/2}} , \qquad (23.8)$$

and

$$A' = [(a_1 - b^*)^2 + a^{*2}]^{1/2} , \qquad (a)$$

$$B' = [(a_2 - b^*)^2 + a^{*2}]^{1/2} , \qquad (b)$$

$$(2.39)$$

$$a^{*2} = -\frac{1}{4} (a_3 - a_3^*)^2 , \qquad (c)$$

$$b^{*2} = \frac{1}{2} (a_3 + a_3^*) . \qquad (d)$$

The variable c_{θ_H} is now related to u^* through the equation

$$c_{\theta_H} = \frac{A_1 + A_2 \text{cn } u^*}{A_3 + A_4 \text{cn } u^*} , \qquad 0 \leq u^* \leq 2K^* . \qquad (2.40)$$

where K^* has the same definition as in Case (i) and k^* is the new modulus of Jacobian elliptic functions and integrals. Since the time rate of change of c_{θ_H} is real, the value of c_{θ_H} must lie between a_1 and a_2. If $a_1 > c_{\theta_H} > a_2$, we have

$$A_1 = a_1 B' + a_2 A' , \qquad (a)$$

$$A_2 = a_2 A' - a_1 B' , \qquad (b)$$

$$(2.41)$$

$$A_3 = A' + B' , \qquad (c)$$

$$A_4 = A' - B' , \qquad (d)$$

and k^{*2} has the value

$$k^{*2} = \frac{(a_1-a_2)^2 - (A'-B')^2}{4A'B'} \quad . \qquad (2.42)$$

If we integrate (2.37) with respect to time t from t_2 to t, we obtain [see (BF259.00)]

$$u^* = \mp \frac{1}{g^*} (A^*)^{1/2} s_{\theta_0} \dot{\Omega} (t-t_2) \quad , \qquad (2.43)$$

where t_2 is the value of t at which $c_{\theta_H} = a_2$. Substituting (2.43) into equation (2.40) we obtain the first-order secular solution for c_{θ_H}. With the time dependence of c_{θ_H} known, equation (2.1) gives the first-order secular solution for ψ_H.

Proceeding as in Case (i), and using equations (2.40) and (2.43) in conjunction with equation (2.19), we find that, to first-order,

$$\frac{c_{\psi_H}}{s_{\theta_H}} dt = - \frac{g^*}{(A^*)^{1/2} s_{\theta_0} \dot{\Omega}} \left[\frac{D_1 A_3}{h(A_3-A_1)} \frac{1+\gamma_3 \text{ cn } u^*}{1+\gamma_1 \text{cn } u^*} \right.$$

$$\left. + \frac{D_2 A_3}{h(A_3+A_1)} \frac{1+\gamma_3 \text{cn } u^*}{1+\gamma_2 \text{cn } u^*} + D_3 \right] du^* \; . \qquad (2.44)$$

where

$$\gamma_1 = \frac{A_4 - A_2}{A_3 - A_1} , \qquad \text{(a)}$$

$$\gamma_2 = \frac{A_4 + A_2}{A_3 + A_1} , \qquad \text{(b)} \qquad (2.45)$$

$$\gamma_3 = \frac{A_4}{A_3} . \qquad \text{(c)}$$

To integrate equation (2.44) with respect to time t, we can use (BF361.62) and rearrange it in the form

$$\frac{c_{\psi_H}}{s_{\theta_H}} dt = - \frac{g^*}{(A^*)^{1/2} s_{\theta^\circ} \dot{\Omega}} \left[\frac{D_1 A_3}{h(A_3 - A_1)} \left(\frac{\gamma_3}{\gamma_1} + \frac{1 - \gamma_3/\gamma_1}{1 - \gamma_1 \operatorname{cn} u^*} \right) \right.$$

$$\left. + \frac{D_2 A_3}{h(A_3 + A_1)} \left(\frac{\gamma_3}{\gamma_2} + \frac{1 - \gamma_3/\gamma_2}{1 + \gamma_2 \operatorname{cn} u^*} \right) + D_3 \right] du^* . \qquad (2.44)$$

If $\gamma_1^2 \neq 1$, and $\gamma_2^2 \neq 1$, we find, from (BF361.54) [or from (BF341.03)], that

$$I_{1i} = \int_{t_i}^{t} \frac{c_{\psi_H}}{s_{\theta_H}} dt$$

$$= - \frac{g^*}{(A^*)^{1/2} s_{\theta^\circ} \dot{\Omega}} \left\{ \left(\frac{D_1 A_3 \gamma_3}{h(A_3 - A_1)\gamma_1} + \frac{D_2 A_3}{h(A_3 + A_1)} \frac{\gamma_3}{\gamma_2} + D_3 \right) u_i^* + \right.$$

$$+ \frac{D_1 A_3}{h(A_3 - A_1)} \quad \frac{\gamma_1 - \gamma_3}{\gamma_1} \left[R_1(u^*, \gamma_1) - R_1(0, \gamma_1) \right]$$

$$+ \frac{D_2 A_3}{h(A_3 + A_1)} \quad \frac{\gamma_2 - \gamma_3}{\gamma_2} \left[R_1(u^*, \gamma_2) - R_1(0, \gamma_2) \right] \Bigg\} \quad , \qquad (2.46)$$

where

$$R_1(u^*, \gamma_i) = \frac{1}{1 - \gamma_i^2} \left[\Pi(u^*, \frac{\gamma_i^2}{\gamma_i^2 - 1}) - \gamma_i \; f_1(u^*, \gamma_i^2) \right] \quad , \quad \text{(a)}$$

$$f_1(u^*, \gamma_i^2) = \left(\frac{1 - \gamma_i^2}{k^{*2} + k'^2 \gamma_i^2} \right)^{1/2} \tan^{-1} \left[\left(\frac{k^{*2} + k^{*\prime 2} \gamma_i^2}{1 - \gamma_i^2} \right)^{1/2} \operatorname{sd} u^* \right] \quad ,$$

$$\text{if} \quad \frac{\gamma_i^2}{\gamma_i^2 - 1} < k^{*2}$$

$$= \operatorname{sd} u^*, \quad \text{if} \quad \frac{\gamma_i^2}{\gamma_i^2 - 1} = k^{*2} \quad ; \qquad \text{(b)} \quad (2.47)$$

$$= \left(\frac{\gamma_i^2 - 1}{k^{*2} + k^{*\prime 2} \gamma_i^2} \right)^{1/2} \ell n \left[\frac{(k^{*2} + k^{*\prime 2} \gamma_i^2)^{1/2} \operatorname{dn} u^* + (\gamma_i^2 - 1)^{1/2} \operatorname{sn} u^*}{(k^{*2} + k^{*\prime 2} \gamma_1^2)^{1/2} \operatorname{dn} u - (\gamma_i^2 - 1)^{1/2} \operatorname{sn} u^*} \right] \quad , \quad \text{if} \quad \frac{\gamma_i^2}{\gamma_i^2 - 1} > k^{*2} \; .$$

If either $\gamma_1^2 = 1$ or $\gamma_2^2 = 1$, the integral in (2.46) is to be replaced with the integral given by either (BF361.51) or (BF341.53).

Equation (2.40) can be used to write

$$I_{21} = \int_{t_2}^{t} c_{\theta_H}^2 \, dt$$

$$= -\frac{g^*}{(A^*)^{1/2} s_{\theta^\circ} \dot{\Omega}} \frac{A_1^2}{A_3^2} \int_0^{u^*} \left[\frac{\gamma_5^2}{\gamma_4^2} - \frac{2\gamma_5(\gamma_4 - \gamma_5)}{\gamma_4^2} \frac{1}{1 + \gamma_4 \operatorname{cn} u^*} + \right.$$

$$\left. + \left(\frac{\gamma_4 - \gamma_5}{\gamma_4} \right)^2 \frac{1}{(1 - \gamma_4 \operatorname{cn} u^*)^2} \right] du^* , \qquad (2.48)$$

where

$$\gamma_4 = \frac{A_4}{A_2} , \qquad (a)$$

$$(2.49)$$

$$\gamma_5 = \frac{A_2}{A_3} . \qquad (b)$$

Integrating with respect to time, we find from (BF341.03) and (BF341.04) that

$$I_2 = -\frac{g^*}{(A^*)^{1/2} s_{\theta^\circ} \dot{\Omega}} \frac{A_1^2}{A_3^2} \left\{ \frac{\gamma_5^2}{\gamma_4^2} u^* + \frac{2\gamma_5(\gamma_4 - \gamma_5)}{\gamma_4^2(1 - \gamma_4^2)} \left[R_1(u^*, \gamma_4^2) - \right. \right.$$

$$- R_1(0, \gamma_4^2)\Big] + \left(\frac{\gamma_4 - \gamma_5}{\gamma_4}\right)^2 \frac{\gamma_4^2(2k^2 - 1) - 2k^2}{(\gamma_4^2 - 1)(k^{*2} - \gamma_4^2 k^{*\prime 2})} \Big[R_1(u^*, \gamma_4^2) -$$

$$- R_1(0, \gamma_4^2)\Big] + 2k^{*2}\Big[R_{-1}(u^*, \gamma_4) - R_{-1}(0, \gamma_4)\Big] + \frac{\gamma_4^2 \, \mathrm{sn}\, u^* \mathrm{dn}\, u^*}{1 - \gamma_4 \mathrm{cn}\, u^*}$$

$$- k^{*2}\Big[R_{-2}(u^*, \gamma_4) - R_{-2}(0, \gamma_4)\Big]\Big\} \qquad (2.50)$$

where

$$R_{-1}(u^*, \gamma_4) = u^* + \frac{\gamma_4}{k^*} \cos^{-1}(\mathrm{dn}\, u^*) \,, \qquad (a)$$

$$(2.51)$$

$$R_{-2}(u^*, \gamma_4) = \frac{1}{k^{*2}}\Big[(k^{*2} - \gamma_4^2 k^{*\prime 2}) u^* + \gamma_4^2 E(u^*) \qquad (b)$$

$$+ 2\gamma_4 k^* \cos^{-1}(\mathrm{dn}\, u^*)\Big] \,,$$

and $\gamma_4^2 \neq 1$.

If $\gamma_4^2 = 1$, the integrals in (2.48) are to be replaced with the integrals given by (BF341.53) and (BF341.54). Thus equations (a),(b),(c),(d),(e),(f) of (2.31) together with (2,40), (2.46) and (2.50) give the associated first-order secular solutions for the six variables.

Case (iv): Two identical real roots and a pair of complex roots

This is a special case of Case (iii). It can be shown in a straightforward manner from (2.40) that if $a_1 = a_2$ then $c_{\theta_H} = a_1$ and θ_H is a constant of the motion. Consequently ψ_H, θ_H, θ' and h are all constants of the motion and ϕ_H and ϕ' are linear functions of time.

BIBLIOGRAPHY

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